Geometry driven collapses for converting a Čech complex into a triangulation of a nicely triangulable shape*

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Abstract

Given a set of points that sample a shape, the Rips complex of the data points is often used in machine-learning to provide an approximation of the shape easily-computed. It has been proved recently that the Rips complex captures the homotopy type of the shape assuming the vertices of the complex meet some mild sampling conditions. Unfortunately, the Rips complex is generally high-dimensional. To remedy this problem, it is tempting to simplify it through a sequence of collapses. Ideally, we would like to end up with a triangulation of the shape. Experiments suggest that, as we simplify the complex by iteratively collapsing faces, it should indeed be possible to avoid entering a dead end such as the famous Bing's house with two rooms. This paper provides a theoretical justification for this empirical observation.

We demonstrate that the Rips complex of a point-cloud (for a well-chosen scale parameter) can always be turned into a simplicial complex homeomorphic to the shape by a sequence of collapses, assuming the shape is nicely triangulable and well-sampled (two concepts we will explain in the paper). To establish our result, we rely on a recent work which gives conditions under which the Rips complex can be converted into a Čech complex by a sequence of collapses. We proceed in two phases. Starting from the Čech complex, we first produce a sequence of collapses that arrives to the Čech complex, restricted by the shape. We then apply a sequence of collapses that transforms the result into the nerve of some "robust" covering of the shape.

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1 Introduction

This paper studies the problem of converting a Čech complex whose vertices sample a shape into a triangulation of that shape using collapses.

Imagine we are given a set of points that sample a shape and we want to build an approximation of the shape from the sample points. A practical approach, often used in machine learning, consists in outputting the Rips complex of the points [8, 13, 12]. Formally, the *Rips complex* of a set of points P at scale α is the abstract simplicial complex whose simplices are subsets of points in P with diameter at most 2α. The Rips complex is an example of a flag complex — the maximal simplicial complex with a given 1-skeleton. As such, it enjoys the property to be completely determined by its 1-skeleton which therefore offers a compact form of storage easy to compute. Recently, precise sampling conditions have been formulated under which the Rips complex reproduces the homotopy type of the shape [9, 3, 5]. Unfortunately, the dimension of the Rips complex can be very large. This suggests a two-phase algorithm for shape reconstruction. The first phase builds the Rips complex of the data points, thus producing an object with the right homotopy type. The second phase simplifies the Rips complex through a sequence of *collapses*. Ideally, we would like to end up with a simplicial complex homeomorphic to the underlying shape.

Yet it is not at all obvious that the Rips complex whose vertices sample a shape contains a subcomplex homeomorphic to that shape. Even if such a subcomplex exists, is there a sequence of collapses that leads to it? Certainly if we want to say anything at all, the geometry of the complex will have to play a key role. As evidence for this, consider a simplicial complex whose vertex set is a noisy point-cloud that samples a 0-dimensional manifold and suppose the complex is composed of a union of Bing's houses with two rooms, one for each connected component in the manifold. Each Bing's house is a 2-dimensional simplicial complex which is contractible but not collapsible. Thus, the complex carries the homotopy type of the 0-dimensional manifold but is not collapsible. Fortunately, it seems that such bad things do not happen in practice, when we start with the Rips complex of a set of points that samples "sufficiently well" a "nice enough" space in $\mathbb{R}^d$. The primary aim of the present work is to understand why. For this, we will focus on the *Čech complex*, a closely related construction. Formally, the Čech complex of a point set P at scale α consists of all simplices spanned by points in P that fit in a ball of radius α. In [5], it was proved that the Rips complex can be reduced to the Čech complex by a sequence of collapses, assuming some sampling conditions are met.

In this paper, we give some mild conditions under which there is a sequence of collapses that converts the Čech complex (and therefore also the Rips complex) into a simplicial complex homeomorphic to the shape. Our result assumes the shape to be *nicely triangulable*, a concept we will explain later in the paper. Perhaps unfortunately, the proof of existence is not very constructive: it starts by sweeping space with offsets of the shape — which are unknown — and builds a sequence of complexes which have no reason to remain close to flag complexes and therefore cannot benefit from the data structure developed in [4]. Nonetheless, even if results presented here do not give yet any practical algorithm, we believe that they provide a better understanding as to why the Čech complex (and therefore the Rips complex) can be simplified by collapses and how this ability is connected to the underlying metric structure of the space.

We now list the principal results of the paper, materialized as brown arrows in Figure 1. In Section 3, we study how the reach of a shape is modified when intersected with a (possibly infinite) collection of small balls. Based on this knowledge, we introduce the *Čech complex restricted by the shape A* in Section 4 and give conditions under which it is homotopy equivalent to A (Theorem 1). In Section 5, we find conditions under which there is a sequence of collapses that goes from the Čech

complex to the restricted Čech complex (Theorem 2). Combined with Theorem 1, this gives an alternative proof to a result [15] recalled in Section 2 (Lemma 2). In Section 6, we define *α-robust coverings* and give conditions under which the restricted Čech complex can be transformed into the nerve of an α-robust covering (Theorem 3). In Section 7, we define and study *nicely triangulable spaces*. Such spaces enjoy the property of having triangulations that can be expressed as the nerve of α-robust coverings for a large range of α. Finally, we exhibit examples of such spaces. Section 8 concludes the paper.

$$\begin{tikzcd}
\mathrm{Rips}(P,\alpha) \arrow[r, rightsquigarrow, "{[5]}"] & \mathrm{Cech}(P,\alpha) \arrow[r, rightsquigarrow, "\text{Theorem 2}"] \arrow[d, leftrightarrow, "\simeq"', "\begin{array}{l}\text{Lemma 1}\\\text{(Nerve Lemma)}\end{array}"] & \mathrm{Cech}_A(P,\alpha) \arrow[r, rightsquigarrow, "\text{Theorem 3}"] \arrow[d, leftrightarrow, "\simeq\ \text{Theorem 1}"] & \text{Nerve of an } \alpha\text{-robust covering of } A \arrow[dl, leftrightarrow, bend left, "\approx?"] \\
 & P^{\oplus\alpha} \arrow[r, hookrightarrow, "\text{Lemma 2 [15]}"] & A &
\end{tikzcd}$$

Figure 1: Logical structure of our results. Brown arrows represent new results. The arrow $\hookrightarrow$ stands for "deformation retracts to". The arrow $\rightsquigarrow$ stands for "can be transformed by a sequence of collapses into". The symbol "$\simeq$" means "homotopy equivalent to" and "$\approx$" means "homeomorphic to".

2 Background

First let us explain some of our terms and introduce the necessary background. $\mathbb{R}^d$ denotes the d-dimensional Euclidean space. $\|x-y\|$ is the Euclidean distance between two points x and y of $\mathbb{R}^d$. The closed ball with center x and radius r is denoted by $\mathrm{B}(x,r)$ and its interior by $\mathrm{B}^\circ(x,r)$. Given a subset $X \subset \mathbb{R}^d$, the α-offset of X is $X^{\oplus\alpha} = \bigcup_{x\in X} B(x,\alpha)$. The Hausdorff distance $d_H(X,Y)$ between the two compact sets X and Y of $\mathbb{R}^d$ is the smallest real number $\varepsilon \geq 0$ such that $X \subset Y^{\oplus\varepsilon}$ and $Y \subset X^{\oplus\varepsilon}$. We write $d(x,Y) = \inf_{y\in Y}\|x-y\|$ for the distance between point $x \in \mathbb{R}^d$ and the set $Y \subset \mathbb{R}^d$ and $d(X,Y) = \inf_{x\in X}\inf_{y\in Y}\|x-y\|$ for the distance between the two sets $X \subset \mathbb{R}^d$ and $Y \subset \mathbb{R}^d$.

A convenient way to build a simplicial complex is to consider the nerve of a collection of sets. Specifically, let $\mathscr{C} = \{C_p \mid p \in P\}$ be a 1-parameter family of sets indexed by $p \in P$. The nerve of the family is the abstract simplicial complex that consists of all non-empty subcollections whose sets have a non-empty common intersection, $\mathrm{Nrv}\,\mathscr{C} = \{\sigma \subset P \mid \sigma \neq \emptyset \text{ and } \bigcap_{p\in\sigma} C_p \neq \emptyset\}$. In this paper, we shall consider nerves of coverings of a shape A. We recall that a *covering* of A is a collection $\mathscr{C} = \{C_p \mid p \in P\}$ of subsets of A so that $A = \bigcup_{p\in P} C_p$. It is a *closed* (*resp. compact*) covering if each set in $\mathscr{C}$ is closed (*resp.* compact). It is a *finite* covering if $\mathscr{C}$ is finite. The Nerve Lemma gives a condition under which the nerve of a covering of a shape shares the topology of the shape. It has several versions [6] and we shall use the following form:

Lemma 1 (Nerve Lemma)**.** *Consider a compact set $A \subset \mathbb{R}^d$. Let $\mathscr{C} = \{C_p \mid p \in P\}$ be a finite closed covering of A. If for every $\emptyset \neq \sigma \subset P$, the intersection $\bigcap_{z\in\sigma} C_z$ is either empty or contractible, then the underlying space of* $\mathrm{Nrv}\,\mathscr{C}$ *is homotopy equivalent to A.*

Hereafter, we shall omit the phrase "the underlying space of" and write $X \simeq Y$ to say that X is homotopy equivalent to Y. Given a finite set of points $P \in \mathbb{R}^d$ and a real number $\alpha \geq 0$, the Čech complex of P at scale α can be defined as $\mathrm{Cech}(P,\alpha) = \mathrm{Nrv}\{B(p,\alpha) \mid p \in P\}$. With this definition and the Nerve Lemma, it is clear that $\mathrm{Cech}(P,\alpha) \simeq P^{\oplus\alpha}$; see the black vertical arrow in Figure 1. Several recent results

have expressed conditions under which $P^{\oplus\alpha}$ recovers the homotopy type of the shape A [15, 11, 10, 5]. Intuitively, the data points P must sample the shape A sufficiently densely and accurately. One of the simplest ways to measure the quality of the sample is to use the reach of the shape. Given a compact subset A of $\mathbb{R}^d$, recall that the *medial axis* $\mathcal{M}_A$ of A is the set of points in $\mathbb{R}^d$ which have at least two closest points in A. The *reach* of A is the infimum of distances between points in A and points in $\mathcal{M}_A$, $\mathrm{Reach}(A) = \inf_{a\in A, m\in\mathcal{M}_A} \|a - m\|$. We have (horizontal black arrow in Figure 1):

Lemma 2 ([15])**.** *Let A and P be two compact subsets of $\mathbb{R}^d$. Suppose there exists a real number ε such that $d_H(A,P) \le \varepsilon < (3-\sqrt{8})\mathrm{Reach}(A)$. Then, $P^{\oplus\alpha}$ deformation retracts to A for $\alpha = (2+\sqrt{2})\varepsilon$.*

Combining Lemma 1 and Lemma 2, we thus get conditions under which $\mathrm{Cech}(P,\alpha) \simeq A$. The next two sections will provide an alternative proof of this result along the way.

Before starting the paper, let us recall what a collapse is. Consider a simplicial complex K and $\sigma_{\min} \in K$. Let Δ be the collection of simplices in K having $\sigma_{\min}$ as a face. Provided that there is a unique inclusion-maximal element $\sigma_{\max} \neq \sigma_{\min}$ in Δ, it is well-known that K deformation retracts to $K \setminus \Delta$ and the operation that removes Δ is called a *collapse*.

3 Reach of spaces restricted by small balls

In this section, we consider a subset $A \subset \mathbb{R}^d$ such that $\mathrm{Reach}(A) > 0$ and prove that as we intersect A with balls of radius $\alpha < \mathrm{Reach}(A)$ the reach of the intersection can only get bigger; see Figure 2. More precisely, let A be a compact subset of $\mathbb{R}^d$ and $\sigma \subset \mathbb{R}^d$. Write $\mathcal{B}(\sigma,\alpha) = \bigcap_{z\in\sigma} B(z,\alpha)$ for the common intersection of balls with radius α centered at σ and assume that $A \cap \mathcal{B}(\sigma,\alpha) \neq \emptyset$. In this section, we establish that $\mathrm{Reach}(A) \le \mathrm{Reach}(A \cap \mathcal{B}(\sigma,\alpha))$ first when σ is reduced to a single point z (Lemma 3), then when σ is finite (Corollary 1) and finally when σ is a compact subset of $\mathbb{R}^d$ (Lemma 5). Although the first generalization is all we need for establishing Theorem 1 in Section 4, the second generalization will turn out to be useful later on in the paper.

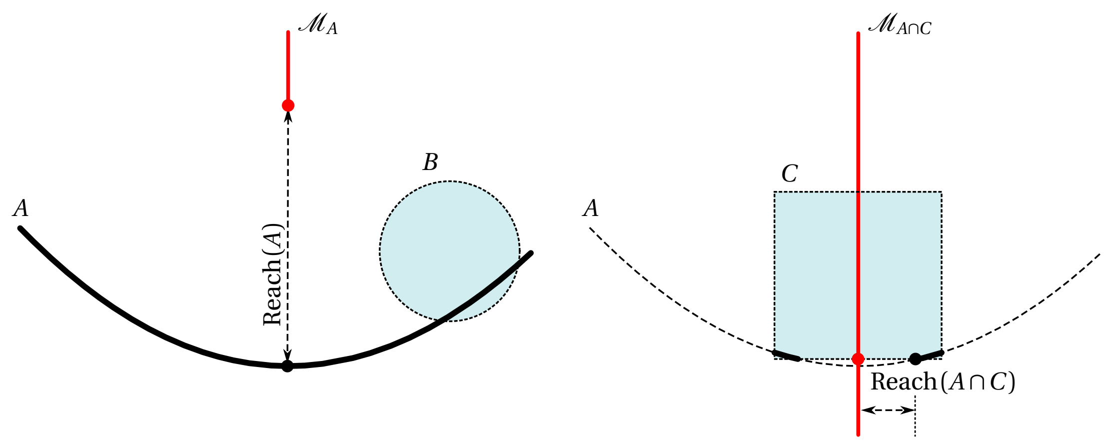

Figure 2: Left: Medial axis $\mathcal{M}_A$ of a shape A. If we intersect A with a ball B whose radius is smaller than the reach of A, the reach of the intersection $A\cap B$ can only get bigger. Right. This property does not hold if we replace the ball B by another set, even with infinite reach such as the cube C.

Lemma 3. *Let $A \subset \mathbb{R}^d$ be a compact set and $B(z,\alpha)$ a closed ball with center z and radius α. If $0 \le \alpha <$ $\mathrm{Reach}(A)$ and $A \cap B(z,\alpha) \neq \emptyset$ then $\mathrm{Reach}(A) \le \mathrm{Reach}(A \cap B(z,\alpha))$.*

Proof. See Figure 3 on the left. Set $\alpha_z = \alpha$ and let $A' = A \cap B(z, \alpha_z)$. Assume, by contradiction, that $\text{Reach}(A') < \text{Reach}(A)$ and consider a point y in the medial axis of A' such that $d(y, A') = \alpha_y < \text{Reach}(A)$. Introduce $A'' = A' \cap B(y, \alpha_y)$ and denote by c and ρ the center and the radius of the smallest ball enclosing A''. Because A'' is contained in both $B(z, \alpha_z)$ and $B(y, \alpha_y)$, the radius of the smallest ball enclosing A'' satisfies $\rho \leq \min\{\alpha_z, \alpha_y\}$. Because $A'' \subset A' \subset A$ we get $d(c, A) \leq d(c, A') \leq d(c, A'') \leq \rho \leq \alpha_y < \text{Reach}(A)$ and therefore c has a unique closest point $\pi_A(c)$ in A. Take r to be any real number such that $\max\{\alpha_z, \alpha_y\} < r < \text{Reach}(A)$. We claim that $r - \sqrt{r^2 - \rho^2} < d(c, A)$; see Figure 3 for a geometric interpretation of the quantity $r - \sqrt{r^2 - \rho^2}$. Indeed, for every $m \in \{z, y\}$, since $A'' \subset B(m, \alpha_m)$ and $\alpha_m \geq \rho$, Lemma 12 (i) implies that the following inclusion holds:

$$\mathrm{B}(c, \alpha_m - \sqrt{\alpha_m^2 - \rho^2}) \;\subset\; \mathrm{B}(m, \alpha_m).$$

Since the map $r \mapsto r - \sqrt{r^2 - \rho^2}$ is strictly decreasing in $[\rho, +\infty)$, we get that

$$\mathrm{B}(c, r - \sqrt{r^2 - \rho^2}) \;\subset\; \mathrm{B}^\circ(z, \alpha_z) \cap \mathrm{B}^\circ(y, \alpha_y).$$

By definition, points of A in $\mathrm{B}^\circ(z, \alpha_z)$ belong to A' and $\mathrm{B}^\circ(y, \alpha_y)$ contains no points of A'. It follows that $\mathrm{B}(c, r - \sqrt{r^2 - \rho^2})$ contains no points of A and $r - \sqrt{r^2 - \rho^2} < d(c, A) = \|c - \pi_A(c)\|$ as claimed. Let us consider the point $x = \pi_A(c) + \frac{r}{d(c,A)}(c - \pi_A(c))$; see Figure 3 on the right. By construction $\|x - \pi_A(c)\| = r < \text{Reach}(A)$ and therefore x has a unique closest point $\pi_A(x) = \pi_A(c)$ in A. Since $\mathrm{B}(c, d(c, A)) \subset \mathrm{B}(x, d(x, A))$, we deduce that

$$\mathrm{B}(c, r - \sqrt{r^2 - \rho^2}) \;\subset\; \mathrm{B}^\circ(x, r).$$

Applying Lemma 12 (ii) we get that $A'' \cap \mathrm{B}^\circ(x, r) \neq \emptyset$ and therefore $\mathrm{B}(x, r)$ contains points of A in its interior. But this contradicts $d(x, A) = r$. □

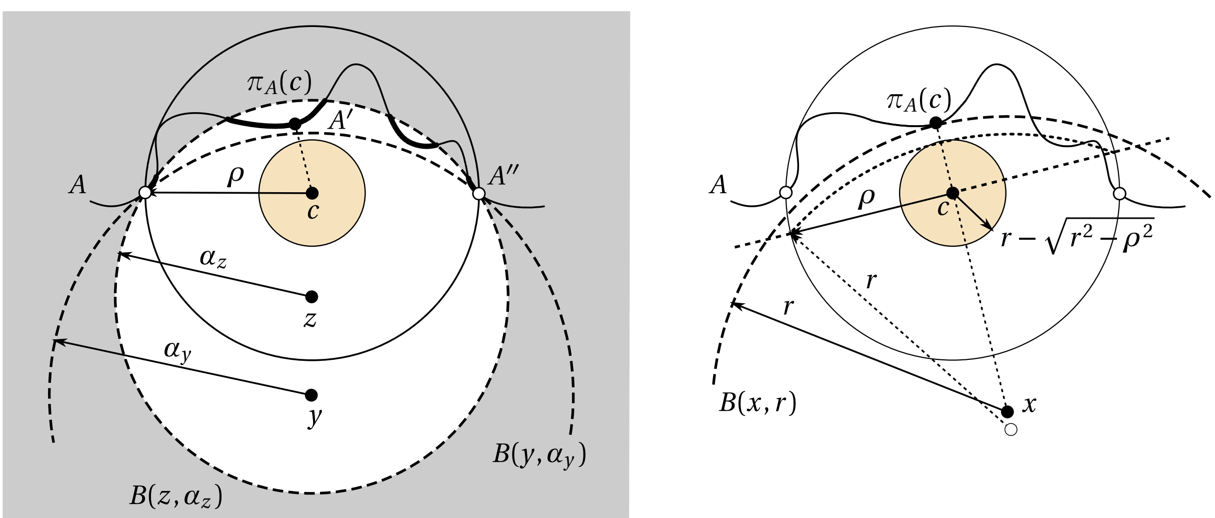

Figure 3: Notation for the proof of Lemma 3. The quantity $r - \sqrt{r^2 - \rho^2}$ represents the height of a spherical cap whose base has radius ρ and which lies on a sphere with radius r.

Corollary 1. *Consider a compact set $A \subset \mathbb{R}^d$ and a finite set $\sigma \subset \mathbb{R}^d$. If $0 \leq \alpha < \mathrm{Reach}(A)$ and $A \cap \mathscr{B}(\sigma, \alpha) \neq \emptyset$ then $\mathrm{Reach}(A) \leq \mathrm{Reach}(A \cap \mathscr{B}(\sigma, \alpha))$.*

Proof. By induction over the size of σ. $\square$

The following lemma is a milestone for the proof of Lemma 5.

Lemma 4. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of non-empty compact subsets of $\mathbb{R}^d$ decreasing with respect of the inclusion order. If there exists a real number r such that $0 \leq r \leq \mathrm{Reach}(A_n)$ for all $n \in \mathbb{N}$, then $r \leq \mathrm{Reach}\left(\bigcap_{n\in\mathbb{N}} A_n\right)$.*

Proof. Letting $A = \bigcap_{n\in\mathbb{N}} A_n$, we first show that the Hausdorff distance $d_H(A_n, A)$ tends to 0 as $n \to +\infty$. For $\varepsilon > 0$, introduce the set $L_\varepsilon = \{x \in \mathbb{R}^d \mid d(x, A) \geq \varepsilon\}$ and notice that $\bigcap_{n\in\mathbb{N}} (L_\varepsilon \cap A_n) = L_\varepsilon \cap A = \emptyset$. Since the sequence $(L_\varepsilon \cap A_n)_{n\in\mathbb{N}}$ is decreasing and consists of compact sets, the only possibility is that $L_\varepsilon \cap A_i = \emptyset$ for some $i \in \mathbb{N}$. Equivalently, $d_H(A_i, A) < \varepsilon$ which proves the convergence of A_n to A under Hausdorff distance.

We now claim the following. For any positive real numbers δ and r' in the open interval $(0, r)$ and any point z' whose distance to A is r', we can find a point c such that

$$A \cap \mathrm{B}(z', r') \subset \mathrm{B}(c, \delta).$$

The claim implies that $A \cap \mathrm{B}(z', r')$ is a singleton and this proves the lemma. We prove the claim by providing a construction for point c; see Figure 4, left. For this, take a real number ε such that:

$$0 < \varepsilon < (r - r')\left(\frac{1}{\sqrt{1 - \left(\frac{\delta}{r}\right)^2}} - 1\right). \tag{1}$$

Since $d_H(A_n, A) \to 0$ as $n \to +\infty$, there is some $i \in \mathbb{N}$ such that $d_H(A_i, A) < \varepsilon$. By hypothesis, $r \leq \mathrm{Reach}(A_i)$. Since $A \subset A_i$, one has $d(z', A_i) \leq d(z', A) = r' < r \leq \mathrm{Reach}(A_i)$ and therefore z' has a unique closest point a_i in A_i. Let us define the point z by

$$z = a_i + \frac{r}{\|z' - a_i\|}(z' - a_i).$$

Since $r \leq \mathrm{Reach}(A_i)$, the open ball $\mathrm{B}^\circ(z, r)$ is disjoint from A_i and since $A \subset A_i$ we obtain that $A \cap \mathrm{B}^\circ(z, r) = \emptyset$ from which we deduce

$$A \cap \mathrm{B}(z', r') \subset \mathrm{B}(z', r') \setminus \mathrm{B}^\circ(z, r).$$

Let c be the centre of the $(d-2)$-sphere S which is the intersection of the two $(d-1)$-spheres $\partial\,\mathrm{B}(z', r')$ and $\partial\,\mathrm{B}(z, r)$. Observe that c lies on line zz'. Take a point $x \in S$ and define $\theta = \angle czx$. It follows from the above inclusion that $A \cap \mathrm{B}(z', r') \subset \mathrm{B}(c, r\sin\theta)$. Hence, to prove the claim it suffices to establish that $r\sin\theta \leq \delta$. Define y as the orthogonal projection of z' on line xz. We have

$$\cos\theta = \frac{\|y - z\|}{\|z' - z\|}.$$

Observe that $\|y - z\| = \|x - z\| - \|x - y\| \geq \|x - z\| - \|x - z'\| = r - r'$. Since $d_H(A, A_i) < \varepsilon$, one has $d(z, A) \leq d(z, A_i) + \varepsilon = r + \varepsilon$ and since $d(z', A) = r'$ one has $\|z' - z\| \leq r + \varepsilon - r'$. This gives using Equation (1)

$$\cos\theta \geq \frac{r - r'}{r + \varepsilon - r'} > \sqrt{1 - \left(\frac{\delta}{r}\right)^2} \tag{2}$$

from which we deduce immediately that $r\sin\theta < \delta$ as required. $\square$

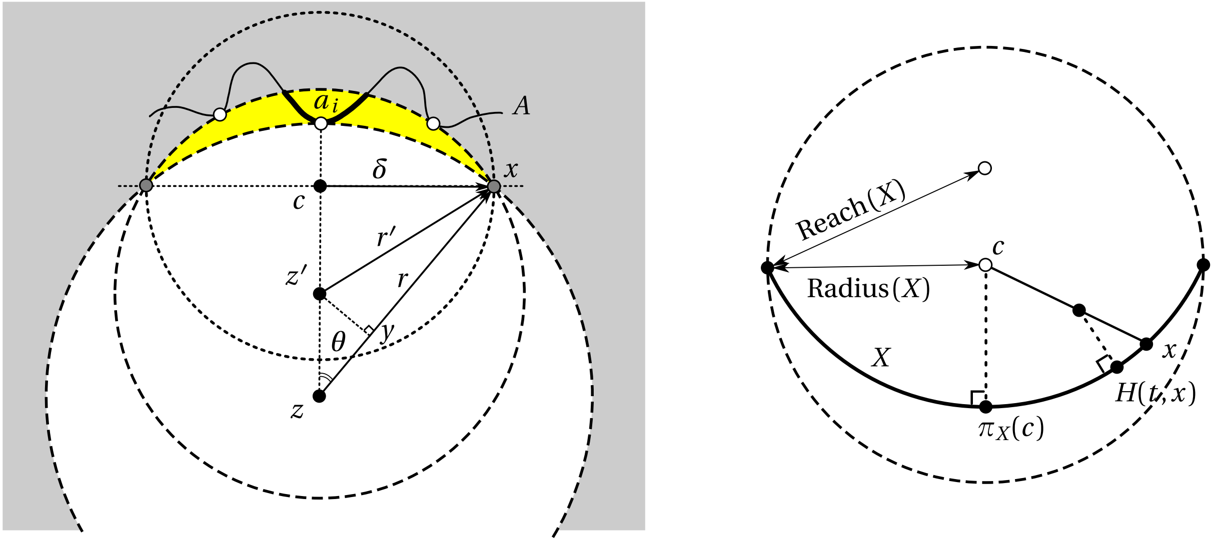

Figure 4: Notation for the proofs of Lemma 4 (left) and Lemma 5 (right).

Lemma 5. *Let A and $\sigma \neq \emptyset$ be two compact sets of $\mathbb{R}^d$. If $0 \leq \alpha < \mathrm{Reach}(A)$ and $A \cap \mathscr{B}(\sigma,\alpha) \neq \emptyset$ then* $\mathrm{Reach}(A) \leq \mathrm{Reach}(A \cap \mathscr{B}(\sigma,\alpha))$.

Proof. Consider a sequence $\{z_i\}_{i\in\mathbb{N}}$ which is dense in σ. Corollary 1 implies that for all $n \geq 0$, we have $\mathrm{Reach}\left(A \cap \bigcap_{i=1}^n \mathrm{B}(z_i,\alpha)\right) \geq \alpha$. Applying Lemma 4 we get that $\mathrm{Reach}\left(A \cap \bigcap_{i\in\mathbb{N}} \mathrm{B}(z_i,\alpha)\right) \geq \alpha$. We claim that $A \cap \bigcap_{i\in\mathbb{N}} \mathrm{B}(z_i,\alpha) = A \cap \bigcap_{z\in\sigma} \mathrm{B}(z,\alpha)$. One direction is trivial. If $a \in A \cap \bigcap_{z\in\sigma} \mathrm{B}(z,\alpha)$, then $a \in A \cap \bigcap_{i\in\mathbb{N}} \mathrm{B}(z_i,\alpha)$. For the other direction, take $a \in A \cap \bigcap_{i\in\mathbb{N}} \mathrm{B}(z_i,\alpha)$ and let us prove that $\forall z \in \sigma, \|z-a\| \leq \alpha$. Assume, by contradiction, that for some $z \in \sigma$, one has $\|z-a\| > \alpha$. Then, there is $i \in \mathbb{N}$ such that $\|z-z_i\| < \|z-a\| - \alpha$, yielding $\|z_i - a\| \geq \|z-a\| - \|z-z_i\| > \alpha$, which is impossible. We have just shown that $a \in A \cap \bigcap_{z\in\sigma} \mathrm{B}(z,\alpha)$. □

4 The restricted Čech complex

Given a subset $A \subset \mathbb{R}^d$, a finite point set P and a real number $\alpha \geq 0$, let us define the Čech complex of P at scale α, $\mathrm{Cech}_A(P,\alpha)$, restricted by A as the set of simplices spanned by points in P that fit in a ball of radius α whose center belongs to A. Equivalently, $\mathrm{Cech}_A(P,\alpha) = \mathrm{Nrv}\{A \cap \mathrm{B}(p,\alpha) \mid p \in P\}$. In this section, we give conditions under which A and $\mathrm{Cech}_A(P,\alpha)$ are homotopy equivalent. Precisely:

Theorem 1. *Let $A \subset \mathbb{R}^d$ be a compact set, $P \subset \mathbb{R}^d$ a finite point set and α a real number such that $0 \leq \alpha < \mathrm{Reach}(A)$ and $A \subset P^{\oplus\alpha}$. Then, $\mathrm{Cech}_A(P,\alpha)$ and A have the same homotopy type.*

Proof. Since $A \subset P^{\oplus\alpha}$, clearly $A = \bigcup_{p\in P}(A \cap B(p,\alpha))$. By Lemma 7 below, for all $\emptyset \neq \sigma \subset P$, the intersection $\bigcap_{z\in\sigma}(A \cap B(z,\alpha))$ is either empty or contractible. We conclude by applying the Nerve Lemma to the collection $\{A \cap B(p,\alpha) \mid p \in P\}$. □

Recall that $\mathscr{B}(\sigma,\alpha) = \bigcap_{z\in\sigma} B(z,\alpha)$ is the common intersection of balls with radius α centered at σ. A key argument in the proof of Theorem 1 is the fact that $A \cap \mathscr{B}(\sigma,\alpha)$ is either empty or contractible, whenever $0 \leq \alpha < \mathrm{Reach}(A)$. To establish this fact, we will need the following lemma. Let $\mathrm{Radius}(X)$ designate the radius of the smallest ball enclosing the compact set X.

Lemma 6. *If $X \subset \mathbb{R}^d$ is a non-empty compact set with* $\mathrm{Radius}(X) < \mathrm{Reach}(X)$*, then X is contractible.*

Proof. We recall that for every point m such that $d(m, X) < \mathrm{Reach}(X)$ there exists a unique point of X closest to m, which we denote by $\pi_X(m)$. Furthermore, we know from [14, page 435] that for $0 < r < \mathrm{Reach}(X)$ the projection map π_X onto X is $\left(\frac{\mathrm{Reach}(X)}{\mathrm{Reach}(X)-r}\right)$-Lipschitz for points at distance less than r from X. Denote by c the center of X; see Figure 4, right. If $x \in X$ and $t \in [0,1]$, one has

$$d((1-t)x + tc, X) \;\leq\; \|(1-t)x + tc - x\| \;\leq\; \|c - x\| \;\leq\; \mathrm{Radius}(X) \;<\; \mathrm{Reach}(X).$$

Thus, the map $H : [0,1] \times X \to X$ defined by $H(t,x) = \pi_X((1-t)x + tc)$ is Lipschitz and defines a deformation retraction of X onto $\{\pi_X(c)\}$. □

We deduce immediately the following lemma. Besides being useful for proving Theorem 1, it will turn out to be a key tool in Section 6.

Lemma 7. *Let A be a compact set of $\mathbb{R}^d$ and α a real number such* $0 \leq \alpha < \mathrm{Reach}(A)$*. For all non-empty compact subsets $\sigma \subset \mathbb{R}^d$, the intersection $A \cap \mathscr{B}(\sigma, \alpha)$ is either empty or contractible.*

Proof. Suppose $A \cap \mathscr{B}(\sigma, \alpha) \neq \emptyset$. By Lemma 5,

$$\mathrm{Radius}(A \cap \mathscr{B}(\sigma, \alpha)) \;\leq\; \alpha \;<\; \mathrm{Reach}(A) \;\leq\; \mathrm{Reach}(A \cap \mathscr{B}(\sigma, \alpha)).$$

By Lemma 6, $A \cap \mathscr{B}(\sigma, \alpha)$ is contractible. □

This lemma can be seen as a variant of Lemma 7 in [1], Proposition 12 in [7] and the local reach lemma in [2] which all say that if A is a k-manifold that intersects a ball B with radius $\alpha < \mathrm{Reach}(A)$, then $A \cap B$ is a topological k-ball.

5 Restricting the Čech complex by collapses

In this section, we state a condition under which there exists a sequence of collapses that transforms $\mathrm{Cech}(P, \alpha)$ into its restricted version $\mathrm{Cech}_A(P, \alpha)$. First we establish several facts about a collection of balls whose centers are close to the shape A and which have a non-empty common intersection that does not intersect A. These facts are formalized in the following lemma. As before, we let $\mathscr{B}(\sigma, \alpha)$ denote the common intersection of balls with radius α centered at σ and by convention, we set $d(A, \emptyset) = +\infty$. Hence, when we write that $d(A, \mathscr{B}(\sigma, \alpha)) = t$ for some $t \in \mathbb{R}$, this implies implicitly that $\mathscr{B}(\sigma, \alpha) \neq \emptyset$.

Lemma 8. *Let $A \subset \mathbb{R}^d$ be a compact set, $\sigma \subset \mathbb{R}^d$ a finite set and $\alpha \geq 0$ such that $d(A, \mathscr{B}(\sigma, \alpha)) = t$ for some $t \in \mathbb{R}$. If $0 < t < \mathrm{Reach}(A) - \alpha$, we have the following properties (see Figure 5, left):*

- *There exists a unique point $x \in \mathscr{B}(\sigma, \alpha)$ whose distance to A is t ;*
- *The set $\sigma_0 = \{p \in \sigma \mid x \in \partial B(p, \alpha)\}$ is non-empty ;*
- $d(A, \mathscr{B}(\sigma_0, \alpha)) = t$.

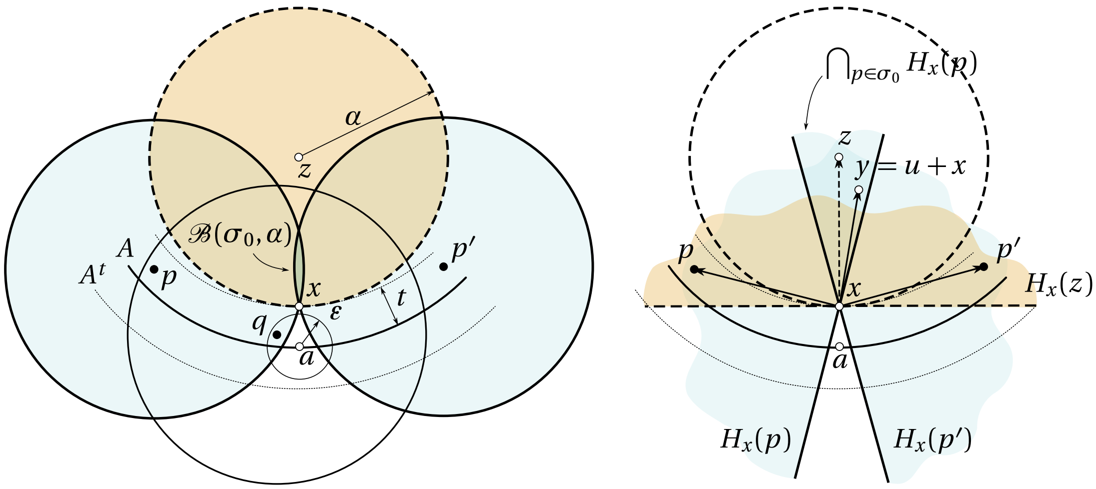

Figure 5: Notation for the proofs of Lemma 8 and Theorem 2. Black dots belong to σ and the ball $B(z,\alpha)$ is instrumental in proving Lemma 8. We show that $\mathscr{B}(\sigma_0,\alpha) \subset B(z,\alpha)$ (on the left) using the fact that $\bigcap_{p\in\sigma_0} H_x(p) \subset H_x(z)$ (on the right), where $H_x(m)$ designates the half-space containing m whose boundary passes through x and is orthogonal to mx.

Proof. Since A and $\mathscr{B}(\sigma,\alpha)$ are both compact sets, there exists at least one pair of points $(a,x) \in A \times \mathscr{B}(\sigma,\alpha)$ such that $\|a-x\| = t$; see Figure 5. By definition, x belongs to $B(p,\alpha)$ for all $p \in \sigma$. Since x lies on the boundary of $\mathscr{B}(\sigma,\alpha)$, it lies on the boundary of $\mathrm{B}(p,\alpha)$ for at least one $p \in \sigma$, showing that $\sigma_0 \neq \emptyset$. Since $\mathrm{Reach}(A^{\oplus t}) = \mathrm{Reach}(A) - t > \alpha$, the point $z = x + \alpha \frac{x-a}{\|x-a\|}$ has a unique closest point x in $A^{\oplus t}$ and the distance $d(A, B(z,\alpha)) = t$ is realized by the pair of points (a,x). To prove that x is the unique point of $\mathscr{B}(\sigma,\alpha)$ whose distance to A is t, it suffices to show that $\mathscr{B}(\sigma,\alpha) \subset \mathrm{B}(z,\alpha)$. Actually, we will prove a stronger result, namely that $\mathscr{B}(\sigma_0,\alpha) \subset \mathrm{B}(z,\alpha)$ which will also imply the third item of the lemma, that is, $d(A, \mathscr{B}(\sigma_0,\alpha)) = t$.

Let us associate to every point $m \in \mathbb{R}^d$ the half-space $H_x(m) = \{u \in \mathbb{R}^d, \langle m-x, u\rangle \geq 0\}$ and observe that for all $m \in \sigma_0 \cup \{z\}$ we have $\|m-x\| = \alpha$; see Figure 5, right. It is easy to check that since x realizes the smallest distance between points of $\mathscr{B}(\sigma,\alpha)$ and a, the following inclusion must hold $\bigcap_{p\in\sigma_0} H_x(p) \subset H_x(z)$. This implies that for all $y \in \bigcap_{p\in\sigma_0} H_x(p)$, the distance between y and the boundary of $H_x(z)$ is always larger than or equal to the distance between y and the boundary of $H_x(p)$ for some $p \in \sigma_0$. Formally, this means that for all $u \in \mathbb{R}^d$ and all $\delta \geq 0$, the following implication holds:

$$\min_{p\in\sigma_0} \langle p-x, u\rangle \geq \delta \implies \langle z-x, u\rangle \geq \delta$$

Plugging $\delta = \frac{\|u\|^2}{2}$ in the above implication and noting that for all $m \in \sigma_0 \cup \{z\}$, the following inequality $2\langle m-x, u\rangle \geq \|u\|^2$ can be rewritten as $\|(m-x)-u\|^2 \leq \alpha^2$, we get that for all $u \in \mathbb{R}^d$, the following implication holds:

$$\max_{p\in\sigma_0} \|p-u-x\|^2 \leq \alpha^2 \implies \|z-u-x\|^2 \leq \alpha^2.$$

Equivalently, $\mathscr{B}(\sigma_0,\alpha) \subset \mathrm{B}(z,\alpha)$, as required. □

Our second theorem (horizontal brown arrow in Figure 1) can be seen as a combinatorial version of Lemma 2.

Theorem 2. *Let $\varepsilon \geq 0$, $\alpha \geq 0$, and $r \geq 0$. Consider a compact set $A \subset \mathbb{R}^d$ with $\mathrm{Reach}(A) \geq r$. Let $P \subset \mathbb{R}^d$ be a finite set such that $d_H(A,P) \leq \varepsilon$. There exists a sequence of collapses from $\mathrm{Cech}(P,\alpha)$ to $\mathrm{Cech}_A(P,\alpha)$ whenever ε, α and r satisfy the following two conditions:*

(i) $\sqrt{2}\alpha < r - \varepsilon$;

(ii) $r - \sqrt{(r-\varepsilon)^2 - \alpha^2} < \alpha - \varepsilon$.

In particular, for $\varepsilon < (3-\sqrt{8})r$ and $\alpha = (2+\sqrt{2})\varepsilon$, conditions (i) and (ii) are fulfilled.

Proof. Letting $\beta = r - \sqrt{(r-\varepsilon)^2 - \alpha^2}$, we observe that condition (i) is equivalent to $\beta < r - \alpha$ assuming $\alpha \geq 0$ and condition (ii) is equivalent to $\beta < \alpha - \varepsilon$. For $t \geq 0$, we define the simplicial complex $K_t = \mathrm{Nrv}\{A^{\oplus t} \cap \mathrm{B}(p,\alpha) \mid p \in P\}$. Notice that $K_0 = \mathrm{Cech}_A(P,\alpha)$ and $K_{+\infty} = \mathrm{Cech}(P,\alpha)$. We prove the theorem in two stages:

(a) First, we prove that K_t does not change as t decreases continuously from $+\infty$ to β. In other words, $K_t = \mathrm{Cech}(P,t)$ for all $t \geq \beta$. Note that this is equivalent to proving that for all non-empty subsets $\sigma \subset P$ and all $t \geq \beta$,

$$\bigcap_{p\in\sigma} \mathrm{B}(p,\alpha) \neq \emptyset \iff A^{\oplus t} \cap \bigcap_{p\in\sigma} \mathrm{B}(p,\alpha) \neq \emptyset.$$

One direction is trivial: if a point belongs to the intersection on the right, then it belongs to the intersection on the left. If the intersection on the left is non-empty, then it contains the center z of the smallest ball enclosing σ and $\mathrm{Radius}(\sigma) \leq \alpha < \sqrt{2}\alpha < r - \varepsilon$. Let us apply Lemma 14 in [3] which says that under these conditions, the convex hull of σ and therefore z cannot be too far away from A, precisely that $z \in \mathrm{Conv}(\sigma) \subset A^{\oplus t}$. Hence, z also belongs to the intersection on the right which therefore is non-empty.

(b) Second, we prove that as t decreases continuously from β to 0, the only changes that may occur in K_t are collapses. Specifically, let Δ_t be the set of simplices that disappear at time t, that is

$$\Delta_t = \{\sigma \subset P \mid d(A, \mathscr{B}(\sigma,\alpha)) = t\}.$$

Suppose $\Delta_t \neq \emptyset$ and let us prove that the deletion of simplices Δ_t from K_t is a collapse for all $t \in (0,\beta]$. Generically, we may assume that the set of simplices Δ_t has a unique inclusion-minimal element $\sigma_{\min}$. We will explain in the appendix how to get rid of this genericity assumption. Since $0 < t \leq \beta < r - \alpha \leq \mathrm{Reach}(A) - \alpha$, Lemma 8 implies that there exists a unique point $x \in \mathscr{B}(\sigma_{\min},\alpha)$ whose distance to A is t; see Figure 5, left. It is easy to see that Δ_t has a unique inclusion-maximal element $\sigma_{\max} = \{p \in P \mid x \in \mathrm{B}(p,\alpha)\}$. Hence, Δ_t consists of all cofaces of $\sigma_{\min}$ and these cofaces are faces of $\sigma_{\max}$. To prove that removing Δ_t from K_t is a collapse, it suffices to establish that $\sigma_{\min} \neq \sigma_{\max}$. By Lemma 8, we know that $\sigma_0 = \{p \in \sigma_{\min} \mid x \in \partial B(p,\alpha)\}$ is non-empty and belongs to Δ_t. By the choice of $\sigma_{\min}$ as the minimal element of Δ_t, we have $\sigma_{\min} \subset \sigma_0$ and therefore x lies on the boundary of $B(p,\alpha)$ for all $p \in \sigma_{\min}$. Since $d(x,A) = t \leq \beta < \mathrm{Reach}(A)$, there exists a unique point $a \in A$ such that $\|a - x\| = t$. Because $d_H(A,P) \leq \varepsilon$, we know that there exists a point $q \in P$ such that $\|q - a\| \leq \varepsilon$. Since $\|q - x\| \leq \|q - a\| + \|a - x\| \leq \varepsilon + t \leq \varepsilon + \beta < \alpha$, we get that x lies in the interior of $B(q,\alpha)$. Therefore, q belongs to $\sigma_{\max}$ but not to $\sigma_{\min}$. Hence, $\sigma_{\min} \neq \sigma_{\max}$. □

6 Collapsing the restricted Čech complex

In this section, we find conditions under which there is a sequence of collapses transforming the restricted Čech complex $\mathrm{Cech}_A(P,\alpha)$ into the nerve of an α-robust covering of A. We start by defining α-robust coverings and state our result in Section 6.1. Our proof technique consists in introducing a 2-parameter family of compact sets $\mathscr{D} = \{D_p(t) \mid (p,t) \in P \times [0,1]\}$ and monitoring its evolution as the parameter t increases continuously from 0 to 1. In Section 6.2, we give some general conditions on $\mathscr{D}$ that guarantee that the simplicial complex $K(t) = \mathrm{Nrv}\{D_p(t) \mid p \in P\}$ only undergoes collapses as t increases from 0 to 1. We believe that these conditions are sufficiently general to be applied to other situations and therefore are interesting in their own right. Armed with this tool, we establish our third result in Section 6.3, that is, we find a family of compact sets $\mathscr{D}$ which enjoys the properties required in Section 6.2 and such that $K(0) = \mathrm{Cech}_A(P,\alpha)$ and $K(1)$ is isomorphic to the nerve of an α-robust covering of A.

6.1 Towards the nerve of α-robust coverings

To state and prove our third theorem, we need some definitions. Given a subset $X \subset \mathbb{R}^d$, we call the intersection of all balls of radius α containing X the *α-hull of X* and denote it by $\mathrm{Hull}_\alpha(X)$. By construction, $\mathrm{Hull}_\alpha(X)$ is convex and $\mathrm{Hull}_{+\infty}(X)$ is the convex hull of X. Setting $\mathrm{Clenchers}_\alpha(X) = \{z \in \mathbb{R}^d \mid X \subset B(z,\alpha)\}$, we have

$$\mathrm{Hull}_\alpha(X) = \bigcap_{z \in \mathrm{Clenchers}_\alpha(X)} B(z,\alpha).$$

Notice that $\mathrm{Clenchers}_\alpha(X)$ is also convex. Indeed, if two balls $B(z_1,\alpha)$ and $B(z_2,\alpha)$ contain X, then any ball $B(\lambda_1 z_1 + \lambda_2 z_2, \alpha)$ with $\lambda_1 + \lambda_2 = 1$, $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ also contains X. Furthermore, if X is compact, so is $\mathrm{Clenchers}_\alpha(X)$.

Definition 1 (α-robust coverings)**.** *A covering $\mathscr{C} = \{C_v \mid v \in V\}$ of A is* α-robust *if (1) each set in $\mathscr{C}$ can be enclosed in an open ball with radius α; (2) $\mathrm{Nrv}\,\mathscr{C} = \mathrm{Nrv}\{A \cap \mathrm{Hull}_\alpha(C_v) \mid v \in V\}$.*

Of course, one may wonder if α-robust coverings of a shape A often arise in practice. Section 7 will address this issue. For now we focus on establishing properties of α-robust coverings.

Lemma 9. *If $\mathscr{C}$ is a finite compact α-robust covering of A and $0 \leq \alpha < \mathrm{Reach}(A)$, then $\mathrm{Nrv}\,\mathscr{C} \simeq A$.*

Proof. We apply the Nerve Lemma to the collection $\{A \cap \mathrm{Hull}_\alpha(C_v) \mid v \in V\}$. Clearly, $A = \bigcup_{v \in V}(A \cap \mathrm{Hull}_\alpha(C_v))$. By Lemma 7, for all $\emptyset \neq \sigma \subset V$, the intersection $A \cap \bigcap_{v \in \sigma} \mathrm{Hull}_\alpha(C_v)$ is either empty or contractible. □

Combining the above lemma and Theorem 1 we thus get that $\mathrm{Nrv}\,\mathscr{C} \simeq \mathrm{Cech}_A(P,\alpha)$ for all finite compact α-robust coverings $\mathscr{C} = \{C_v \mid v \in V\}$ of A with $0 \leq \alpha < \mathrm{Reach}(A)$. Next theorem strengthens this result and states mild conditions on P and V under which there exists a sequence of collapses transforming $\mathrm{Cech}_A(P,\alpha)$ into a simplicial complex isomorphic to $\mathrm{Nrv}\,\mathscr{C}$.

Theorem 3. *Let A be a compact set of $\mathbb{R}^d$ and α a real number such that $0 \leq \alpha < \mathrm{Reach}(A)$. Let $\mathscr{C} = \{C_v \mid v \in V\}$ be a compact α-robust covering of A. Let P be a finite point set and suppose there exists an injective map $f : V \to P$ such that $C_v \subset \mathrm{B}^\circ(f(v),\alpha)$ for all $v \in V$. Then, there exists a sequence of collapses from $\mathrm{Cech}_A(P,\alpha)$ to $f(\mathrm{Nrv}\,\mathscr{C}) = \{f(\sigma) \mid \sigma \in \mathrm{Nrv}\,\mathscr{C}\}$.*

The proof is given in Section 6.3.

6.2 Evolving families of compact sets

In this section, we present a tool that will be useful in Section 6.3 for establishing Theorem 3. Consider a covering of a topological space A and suppose this covering evolves over time. We state conditions under which the evolution of the nerve of this covering only undergoes collapses. Conditions are formulated in a very general setting. We only require A to be a compact topological space: no need for A to possess an Euclidean structure nor a reach. We do not even need to endow A with a metric structure. Before stating our lemma, let us introduce one additional piece of notation. Given a finite set σ and a map $s : \sigma \to [0,1)$, we write $s' \succ s$ to designate a map $s' : \sigma \to [0,1]$ such that $s'(p) > s(p)$ for all $p \in \sigma$. We will say that the map s is constant if $s(p) = s(q)$ for all $(p,q) \in \sigma^2$.

Lemma 10. *Let A be a compact topological space and P a finite set. Consider a 2-parameter family of compact subsets of A, $\mathscr{D} = \{D_p(t) \mid (p,t) \in P \times [0,1]\}$, and suppose the following four properties are satisfied:*

(a) *For all $0 \le t < t' \le 1$ and all $p \in P$, we have $D_p(t') \subset D_p(t)^\circ$;*

(b) $\bigcup_{p\in P} D_p(1) = A$;

(c) *For all $\emptyset \neq \sigma \subset P$ and all maps $s : \sigma \to [0,1]$, the intersection $\mathscr{D}(\sigma,s) = \bigcap_{p\in\sigma} D_p(s(p))$ is either empty or connected;*

(d) *For all $\emptyset \neq \sigma \subset P$ and all maps $s : \sigma \to [0,1)$, the following implication holds: $\mathscr{D}(\sigma,s) \neq \emptyset$ and $\mathscr{D}(\sigma,s') = \emptyset$ for all $s' \succ s$ implies that $\mathscr{D}(\sigma,s)$ is reduced to a single point.*

Then, there exists a sequence of collapses which transforms $\mathrm{Nrv}\{D_p(0) \mid p \in P\}$ *into* $\mathrm{Nrv}\{D_p(1) \mid p \in P\}$.

Proof. The nerve $K_t = \mathrm{Nrv}\{D_p(t) \mid p \in P\}$ changes while t increases continuously from 0 to 1. Because of condition (a), the only changes that may occur are the disappearances of some simplices. Let Δ_t be the set of simplices that disappear at time t. Suppose $\Delta_t \neq \emptyset$ and let us prove that the deletion of simplices Δ_t from K_t is a collapse for all $t \in [0,1)$. Generically, we may assume that the set of simplices Δ_t has a unique inclusion-minimal element $\sigma_{\min}$. We will explain in the appendix how to get rid of this genericity assumption. Assuming we are in this generic situation, we do not need anymore conditions (c) and (d) but can replace them with the weaker conditions (c') and (d') obtained by considering constant maps for s and s'. Since $\bigcap_{p\in\sigma_{\min}} D_p(t) \neq \emptyset$ and $\bigcap_{p\in\sigma_{\min}} D_p(t+\eta) = \emptyset$ for all $0 < \eta \le 1-t$, condition (d') implies that $\bigcap_{p\in\sigma_{\min}} D_p(t) = \{a\}$ for some $a \in A$. It is easy to see that Δ_t has a unique inclusion-maximal element $\sigma_{\max} = \{p \in P \mid a \in D_p(t)\}$. Hence, Δ_t consists of all cofaces of $\sigma_{\min}$ and these cofaces are faces of $\sigma_{\max}$. To prove that removing Δ_t from K_t is a collapse, it suffices to establish that $\sigma_{\min} \neq \sigma_{\max}$. We proceed in two steps:

Step 1: Let us prove that a lies on the boundary of $D_p(t)$ for all $p \in \sigma_{\min}$. For this, define $\sigma_0 = \{p \in \sigma_{\min} \mid a \in \partial D_p(t)\}$. We start by establishing that $\sigma_0 \neq \emptyset$. Indeed, suppose for a contradiction that a belongs to the interior of $D_p(t)$ for all $p \in \sigma_{\min}$. This implies that for some $\eta > 0$, we have $B(a,\eta) \subset \bigcap_{p\in\sigma_{\min}} D_p(t) = \{a\}$, yielding a contradiction. Let us now prove that $\sigma_0 = \sigma_{\min}$. Suppose for a contradiction that σ_0 is a proper subset of $\sigma_{\min}$. Consider $\eta > 0$ such that $B(a,\eta) \subset D_p(t)$ for all $p \in \sigma_{\min} \setminus \sigma_0$. We have

$$a \in \bigcap_{p\in\sigma_0} D_p(t) \cap B(a,\eta) \subset \bigcap_{p\in\sigma_{\min}} D_p(t) = \{a\}.$$

and therefore $\bigcap_{p\in\sigma_0} D_p(t)$ is contained in $\{a\} \cup (\mathbb{R}^d \setminus B(a,\eta))$ which consists of two connected components. The only possibility for (c') to hold is that $\bigcap_{p\in\sigma_0} D_p(t) = \{a\}$. Because of (a), for all $t < t' \le 1$,

we get that $\bigcap_{p\in\sigma_0} D_p(t') \subset \bigcap_{p\in\sigma_0} D_p(t)^\circ = \emptyset$. It follows that σ_0 disappears at time t and the minimality of $\sigma_{\min}$ implies that $\sigma_0 = \sigma_{\min}$, yielding a contradiction. Thus, a lies on the boundary of $D_p(t)$ for all $p \in \sigma_{\min}$.

Step 2: Let us prove that $\sigma_{\max} \neq \sigma_{\min}$. By condition (b), we have $a \in A = \bigcup_{p\in P} D_p(1)$ and therefore a belongs to $D_q(1)$ for some $q \in P$. Since $t < 1$, condition (a) implies that $a \in D_q(1) \subset D_q(t)^\circ$ and therefore $q \in \sigma_{\max}$. On the other hand, $a \notin \partial D_q(t)$ and therefore $q \notin \sigma_{\min}$. It follows that $\sigma_{\max} \neq \sigma_{\min}$. □

Remark. Somewhat surprisingly, condition (c) of Lemma 10 is weaker than the condition required by the Nerve Lemma for guaranteeing that the simplicial complex $K_t = \mathrm{Nrv}\{D_p(t) \mid p \in P\}$ is homotopy equivalent to A at some particular value of $t \in [0,1]$. In particular, if the Nerve Lemma applies at time $t = 0$, that is, if $\bigcap_{p\in\sigma} D_p(0)$ is either empty or contractible for all $\emptyset \neq \sigma \subset P$ and if furthermore the four conditions of Lemma 10 hold, then K_t will have the right homotopy type for all $t \in [0,1]$.

6.3 Final

For proving Theorem 3, we build a 2-parameter family of compact sets $\mathscr{D} = \{D_p(t) \mid (p,t) \in P \times [0,1]\}$ in such a way that if we let $K(t) = \mathrm{Nrv}\{D_p(t) \mid p \in P\}$, then $\mathrm{Cech}_A(P,\alpha) = K(0)$ and $f(\mathrm{Nrv}\,\mathscr{C}) = K(1)$. We then prove that this family meets the hypotheses of Lemma 10, implying that $\mathrm{Cech}_A(P,\alpha)$ can be transformed into $f(\mathrm{Nrv}\,\mathscr{C})$ by a sequence of collapses obtained by increasing continuously t from 0 to 1. To define the family $\mathscr{D}$, let us first associate to every point $p \in P$ the set

$$Z(p) = \begin{cases} \mathrm{Clenchers}_\alpha(C_v) & \text{if } f^{-1}(p) = \{v\}, \\ \{p^+, p^-\} & \text{if } f^{-1}(p) = \emptyset, \end{cases}$$

where p^+ and p^- are two points which are symmetric with respect to p and chosen such that $B(p^+,\alpha) \cap B(p^-,\alpha) = \emptyset$. We then set

$$D_p(t) = A \cap \bigcap_{z\in Z(p)} \mathrm{B}((1-t)p + tz, \alpha).$$

Let us first check that $\mathrm{Cech}_A(P,\alpha) = K(0)$ and $f(\mathrm{Nrv}\,\mathscr{C}) = K(1)$. We claim that $Z(p) \neq \emptyset$ for all $p \in P$. Let us consider two cases. First, if $f^{-1}(p) = \emptyset$, then by definition $Z(p) = \{p^+, p^-\} \neq \emptyset$. Second, if $f^{-1}(p) = \{v\}$, then $Z(p)$ contains at least p since $C_v \subset \mathrm{B}^\circ(p,\alpha)$ by hypothesis. Thus, $D_p(0) = A \cap \bigcap_{z\in Z(p)} B(p,\alpha) = A \cap B(p,\alpha)$ and $K(0) = \mathrm{Cech}_A(P,\alpha)$. On the other hand, we have $D_p(1) = A \cap \bigcap_{z\in Z(p)} B(z,\alpha)$ which we can rewrite as

$$D_p(1) = \begin{cases} A \cap \mathrm{Hull}_\alpha(C_v) & \text{if } f^{-1}(p) = \{v\}, \\ \emptyset & \text{if } f^{-1}(p) = \emptyset. \end{cases}$$

Thus, $K(1) = f(\mathrm{Nrv}\,\mathscr{C})$. By construction, each cell $D_p(t)$ is the intersection of the shape with a collection of balls whose center set $Z(p)$ is either empty or compact and the collection of cells $\{D_p(t) \mid p \in P\}$ cover the shape. Thus, combining Lemma 7 and the Nerve Lemma, we get that $\mathrm{Nrv}\,K(t) \simeq A$ for all $t \in [0,1]$. We are now ready to prove a stronger result, namely that as t increases continuously from 0 to 1, the only changes that may occur in $K(t)$ are collapses.

Proof of Theorem 3. It suffices to establish that the family $\mathscr{D}$ defined above satisfies conditions (a), (b), (c) and (d) of Lemma 10.

(a) Let us prove that for all $0 \leq t < t' \leq 1$ and all $p \in P$, we have $D_p(t') \subset D_p(t)^\circ$. If $f^{-1}(p) = \emptyset$, this is easy to see. Suppose $f^{-1}(p) = \{v\}$; see Figure 6, left. Since $C_v \subset \mathrm{B}^\circ(p,\alpha)$, we can find $\delta > 0$ such that for every point x in $B(p,\delta)$, the inclusion $C_v \subset \mathrm{B}(x,\alpha)$ holds. Equivalently, $B(p,\delta) \subset \mathrm{Clenchers}_\alpha(C_v)$. Consider the half-line L_u emanating from p in direction u. Because $\mathrm{Clenchers}_\alpha(C_v)$ is convex and compact, it intersects the half-line L_u in a closed line segment pz_u with $\|p - z_u\| \geq \delta$. Since, for every point x on the closed line segment $z_{-u}z_u$, we have $B(z_u,\alpha) \cap B(z_{-u},\alpha) \subset B(x,\alpha)$, it follows that one can rewrite $D_p(t)$ as

$$D_p(t) = A \cap \bigcap_{u \in \mathbb{S}^{d-1}} B((1-t)p + tz_u, \alpha).$$

Write $p_u^t = (1-t)p + tz_u$ for short and observe that $\|p_u^{t'} - p_u^t\| = (t'-t)\|p - z_u\| \geq (t'-t)\delta$. Hence, as we move from z_{-u} to z_u on the line segment $z_{-u}z_u$, we meet the five points $p_{-u}^{t'}$, p_{-u}^t, p, p_u^t and $p_u^{t'}$ in this order and the distance between the first two points and the last two points in the sequence is lower bounded by $(t'-t)\delta$. We will now use the following observation. If we consider a point x on the line segment ab such that $\|a - x\| \geq \varepsilon$ and $\|b - x\| \geq \varepsilon$, then $B(a,\alpha) \cap B(b,\alpha) \subset B(x, \sqrt{\alpha^2 - \varepsilon^2})$. Setting $\alpha_0 = \sqrt{\alpha^2 - (t'-t)^2\delta^2}$, we thus get that $B(p_{-u}^{t'},\alpha) \cap B(p_u^{t'},\alpha) \subset B(p_{-u}^t,\alpha_0) \cap B(p_u^t,\alpha_0)$ and since $\alpha_0 < \alpha$

$$D_p(t') \;\subset\; A \cap \bigcap_{u \in \mathbb{S}^{d-1}} B((1-t)p + tz_u, \alpha_0) \;\subset\; D_p(t)^\circ.$$

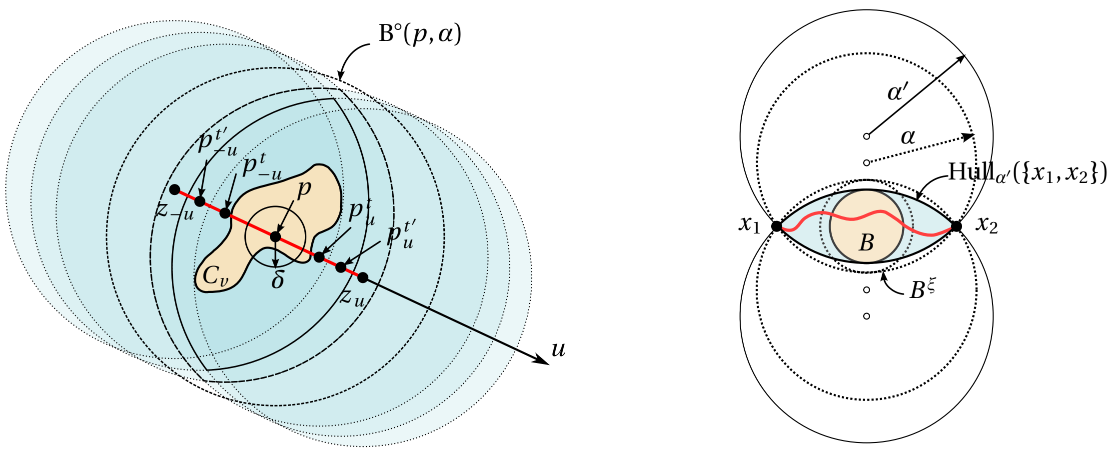

Figure 6: Notation for the proof of Theorem 3.

(b) Clearly, $\bigcup_{p \in P} D_p(1) = A$.

(c) Given $\sigma \subset P$ and a map $s : \sigma \to [0,1]$, we introduce the intersection of balls with radius α

$$\mathscr{D}(\sigma, s) = \bigcap_{p \in \sigma} D_p(s(p)) = A \cap \bigcap_{p \in \sigma} \bigcap_{z \in Z(p)} B((1 - s(p))p + s(p)z, \alpha).$$

By Lemma 7, the intersection $\mathscr{D}(\sigma, s)$ is either empty or connected.

(d) Consider $\sigma \subset P$ and a map $s : \sigma \to [0,1)$ such that $\mathscr{D}(\sigma,s) \neq \emptyset$. Let us prove that if $\mathscr{D}(\sigma,s') = \emptyset$ for all maps $s' : \sigma \to [0,1]$ with $s' \succ s$, then $\mathscr{D}(\sigma,s)$ is a singleton. Assume, by contradiction, that $\mathscr{D}(\sigma,s)$ contains two points x_1 and x_2 and let us prove that we can find a map $s' : \sigma \to [0,1]$ such that $s' \succ s$ and $\mathscr{D}(\sigma,s') \neq \emptyset$. Take α' such that $\alpha < \alpha' < \mathrm{Reach}(A)$. Since $A \cap \mathrm{Hull}_{\alpha'}(\{x_1,x_2\})$ contains both x_1 and

x_2, it is non-empty and therefore contractible by Lemma 7; see Figure 6, right. In particular, there is a path connecting the points x_1 and x_2 in $A \cap \mathrm{Hull}_{\alpha'}(\{x_1, x_2\})$. This path has to intersect the largest ball B contained in $\mathrm{Hull}_{\alpha'}(\{x_1, x_2\})$ and therefore $A \cap B \neq \emptyset$. For $\xi > 0$ sufficiently small we have

$$A \cap B^{\oplus \xi} \ \subset\ A \cap \mathrm{Hull}_{\alpha}(\{x_1, x_2\}) \ \subset\ \mathscr{D}(\sigma, s).$$

By moving slightly the centers of the balls defining $\mathscr{D}(\sigma, s)$, that is, by replacing the map s by a map $s' \succ s$ such that $s'(p) - s(p)$ is small enough for all $p \in \sigma$, we get a new set $\mathscr{D}(\sigma, s')$ that still contains B. Since $\emptyset \neq A \cap B$, we thus get that $\mathscr{D}(\sigma, s') \neq \emptyset$, reaching a contradiction. $\square$

7 Nicely triangulable spaces

Given a space A and a sample P of A, we are seeking a sequence of collapses that transform the Čech complex of P with scale parameter α into a *triangulation* of A. We recall that a *triangulation* of A is a simplicial complex whose underlying space is homeomorphic to A. If A has a triangulation, then A is said to be *triangulable*. In particular, we know that smooth manifolds are triangulable [16]. Unfortunately, the proof involves barycentric subdivisions whose dual meshes are not likely to have convex cells and therefore have little chance of being α-robust coverings. And yet, we know that if a triangulation T of a space A is the nerve of an α-robust covering of A, then the previous section provides conditions under which $\mathrm{Cech}_A(P, \alpha)$ can be transformed into T by a sequence of collapses. This raises the question of whether, given a space A and a scale parameter α, it is possible to find a triangulation T of A which is the nerve of some α-robust covering of A. In this section, we focus on the question and exhibit spaces enjoying this property.

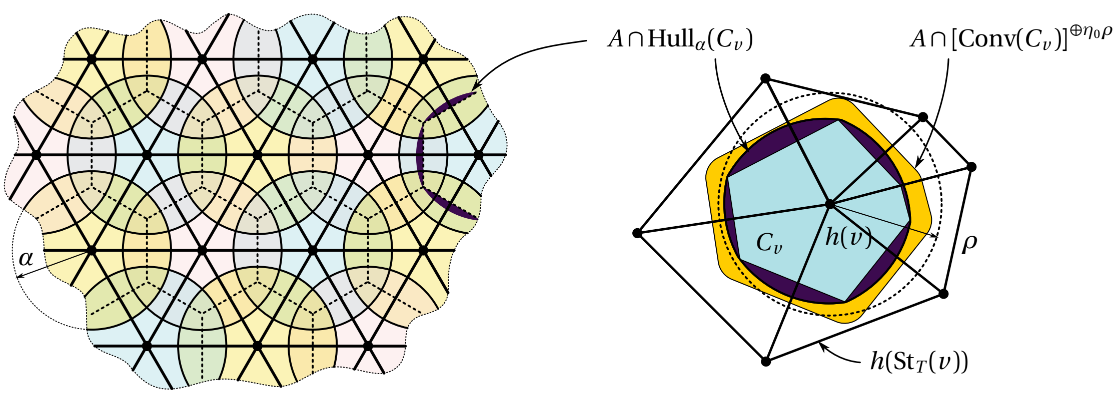

Figure 7: Left: the collection of disks and the collection of Voronoi regions both form an α-robust covering of the plane. Right: A triangulation is nice in our context when, among other things, it is the nerve of a collection of cells C_v with size ρ such that $A \cap [\mathrm{Conv}(C_v)]^{\oplus \eta_0 \rho} \subset h(\mathrm{St}_T(v))$ for some $\eta_0 > 0$. This property will be preserved by $C^{1,1}$ diffeomorphisms for ρ small enough.

As a warm-up, we study the case $A = \mathbb{R}^2$; see Figure 7. Consider a Delaunay triangulation T of $\mathbb{R}^2$ with vertex set V and write $C_v = \{x \in \mathbb{R}^2 \mid \|x - v\| \leq \|x - u\| \text{ for all } u \in V\}$ for the Voronoi cell of $v \in V$. Setting $\mathscr{C} = \{C_v \mid v \in V\}$ for the collection of Voronoi cells, we have that $T = \mathrm{Nrv}\,\mathscr{C}$. If all angles in T are acute, then the Voronoi cell C_v is contained in the star of v and so is $\mathrm{Hull}_\alpha(C_v)$ for α large enough. In particular, by choosing carefully V and α, we can ensure that $\mathscr{C}$ is an α-robust covering of the plane. Writing $\mathrm{St}_T(v)$ for the set of simplices of T incident to v and $\mathrm{Conv}(X) = \mathrm{Hull}_{+\infty}(X)$ for the convex hull of X, this motivates the following definition.

Definition 2 (nice triangulation)**.** *Let ρ and δ be two positive real numbers. A triangulation T of $A \subset \mathbb{R}^d$ is said to be* (ρ,δ)-nice *with respect to $(h,\mathscr{C})$ if h is a homeomorphism from $|T|$ to A, $\mathscr{C} = \{C_v \mid v \in V\}$ is a finite compact covering of A such that $\operatorname{Nrv}\mathscr{C} = T$ and the following conditions hold:*

(i) $h(\sigma) \subset \bigcup_{v\in\sigma} C_v$ *for all simplices* $\sigma \in T$*;*

(ii) $C_v \subset \mathrm{B}^\circ(h(v),\rho)$ *for all* $v \in V$*;*

(iii) $A \cap [\operatorname{Conv}(C_v)]^{\oplus\delta} \subset h(\operatorname{St}_T(v))$ *for all* $v \in V$.

If T is a (ρ,δ)-nice triangulation of A with respect to $(h,\mathscr{C})$, then the next lemma gives conditions on α, ρ and δ which guarantee that $\mathscr{C}$ is an α-robust covering of A.

Lemma 11. *Let A be a compact set of $\mathbb{R}^d$ and suppose T is a (ρ,δ)-nice triangulation of A with respect to $(h,\mathscr{C})$. Then $\mathscr{C}$ is an α-robust covering of A whenever the following two conditions are fulfilled: (1) $\rho \le \alpha$ and (2) $\alpha - \sqrt{\alpha^2-\rho^2} \le \delta$.*

Proof. Suppose $\mathscr{C} = \{C_v \mid v \in V\}$ and let $v \in V$. By Lemma 13, $\operatorname{Hull}_\alpha(C_v) \subset [\operatorname{Conv}(C_v)]^{\oplus\delta}$; see Figure 7, right. It follows that $C_v \subset A \cap \operatorname{Hull}_\alpha(C_v) \subset h(\operatorname{St}_T(v))$ from which we deduce the sequence of inclusions

$$T = \operatorname{Nrv}\mathscr{C} \subset \operatorname{Nrv}\{A \cap \operatorname{Hull}_\alpha(C_v) \mid v \in V\} \subset \operatorname{Nrv}\{h(\operatorname{St}_T(v)) \mid v \in V\} = T.$$

The nerves on the left and on the right are equal, showing that $\operatorname{Nrv}\mathscr{C} = \operatorname{Nrv}\{A \cap \operatorname{Hull}_\alpha(C_v) \mid v \in V\}$. □

Observe that if T is a $(\rho,\eta_0\rho)$-nice triangulation of A for some $\eta_0 > 0$, then conditions (1) and (2) of Lemma 11 are satisfied for $\delta = \eta_0\rho$ as soon as ρ is small enough. Of course, the difficult question is whether such a triangulation T can always be found for arbitrarily small ρ.

Definition 3 (nicely triangulable)**.** *We say that $A \subset \mathbb{R}^d$ is nicely triangulable if we can find $\rho_0 > 0$ and $\eta_0 > 0$ such that for all $0 < \rho < \rho_0$, there is a $(\rho,\eta_0\rho)$-nice triangulation of A.*

Theorem 4. *Suppose $A \subset \mathbb{R}^d$ is nicely triangulable. For every $0 < \alpha < \operatorname{Reach}(A)$, there exists $\varepsilon_0 > 0$ such that for all finite point set $P \subset \mathbb{R}^d$ and all $0 < \varepsilon < \varepsilon_0$ satisfying $A \subset P^{\oplus\varepsilon}$, the complex $\operatorname{Cech}_A(P,\alpha)$ can be transformed into a triangulation of A by a sequence of collapses.*

Proof. By definition, we can find $\rho_0 > 0$ and $\eta_0 > 0$ such that for all $0 < \rho < \rho_0$, there is a $(\rho,\eta_0\rho)$-nice triangulation T of A with respect to $(h,\mathscr{C})$. Let us choose ρ small enough so that $\rho < \alpha$ and $\alpha - \sqrt{\alpha^2-\rho^2} \le \eta_0\rho$. Lemma 11 then implies that $\mathscr{C}$ is a compact α-robust covering of A. Set $e(T,h) = \frac{1}{2}\inf\|h(v_1) - h(v_2)\|$ where the infimum is over all pairs of vertices $v_1 \neq v_2$ of T and let ε_0 be the minimum of $e(T,h)$ and $\alpha - \rho$. Consider a function $f : \operatorname{Vert}(T) \to P$ that maps each vertex v to a point of P closest to $h(v)$. Note that f is injective, $\|h(v) - f(v)\| \le \varepsilon$ and $C_v \subset \mathrm{B}^\circ(f(v),\alpha)$ for all $v \in V$. Applying Theorem 3 yields the existence of a sequence of collapses from $\operatorname{Cech}_A(P,\alpha)$ to $f(T)$. □

We now establish that the property of being nicely triangulable is preserved by $C^{1,1}$ diffeomorphisms between manifolds.

Theorem 5. *Let M and M' be two compact $C^{1,1}$ k-manifolds embedded respectively in $\mathbb{R}^d$ and $\mathbb{R}^{d'}$ and $\Phi : M \to M'$ a $C^{1,1}$ diffeomorphism. M is nicely triangulable if and only if M' is nicely triangulable.*

The proof is given in the Appendix. We conclude the paper with a theorem that provides a few templates of nicely triangulable manifolds. Thanks to Theorem 5 the property for a template to be nicely triangulable is inherited by all manifolds $C^{1,1}$ diffeomorphic to it or equivalently by all manifolds with a positive reach homeomorphic to it [14].

Theorem 6. *The following embedded manifolds are nicely triangulable:*

1. *The unit 2-sphere* $\mathbb{S}^2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sum_{i=1}^3 x_i^2 = 1\}$*;*
2. *The flat torus* $\mathbb{T}^2 = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 = 1$ *and* $x_3^2 + x_4^2 = 1\}$*;*
3. *The m-dimensional Euclidean space $\mathbb{R}^m$, embedded in $\mathbb{R}^d$ for some $m \le d$.*

Proof. For $A \in \{\mathbb{S}^2, \mathbb{T}^2, \mathbb{R}^m\}$, we proceed as follows. We build a triangulation T parameterized by some integer n and consider a map $h : |T| \to A$. The integer n will control the size of elements in $h(T)$: the larger n the smaller the image of simplices under h. We then consider the barycentric subdivision K of T and associate to each vertex v of T the cell $C_v = \bigcup_{\sigma \in \mathrm{St}_K(v)} h(\sigma)$. The collection of cells C_v forms a covering $\mathscr{C}$ of A. In the three cases, it is not difficult to see that we can find $\eta_0 > 0$ such that T is $(\rho, \eta_0\rho)$-nice with respect to $(h, \mathscr{C})$ for some $\rho > 0$. Furthermore, the value of ρ can be made as small as desired by increasing n. We thus conclude that A is nicely triangulable. Below, we just describe how T and h are chosen in each case.

1. $\mathbb{S}^2$ **is nicely triangulable.** We start with an icosahedron centered at the origin and subdivide each triangular face into 4^n equilateral triangles. The resulting triangulation T is then projected onto the sphere, using the projection map $h : |T| \to \mathbb{S}^2$ defined by $h(x) = \frac{x}{\|x\|}$. Let V be the vertex set of T. Notice that all vertices $v \in V$ have degree 6 but the 12 vertices in the original icosahedron which have degree 5. As n increases, the star of each vertex v projects onto the sphere in a set that becomes arbitrarily close to a regular planar k-gon centered at v where k is the degree of v.

2. $\mathbb{T}^2$ **is nicely triangulable.** The map $H : \mathbb{R}^2 \to \mathbb{T}^2$ defined by $H(s,t) = (\cos s, \sin s, \cos t, \sin t)$ is locally isometric and its restriction $h : [0, 2\pi)^2 \to \mathbb{T}^2$ is an homeomorphism. The idea is to build a periodic tiling of $\mathbb{R}^2$ made up of identical isosceles triangles as in Figure 8, left. Let a and b be the respective height and basis of the triangles. Consider two integers n and k such that $na = kb = 2\pi$. Taking $k = \lfloor \frac{\sqrt{3}}{2} n \rfloor$ we get that the ratio $\frac{a}{b}$ tends to $\frac{\sqrt{3}}{2}$ as $n \to +\infty$. Thus, the map H turns the periodic tiling of $\mathbb{R}^2$ into a triangulation of $\mathbb{T}^2$ whose triangles become arbitrarily close to equilateral triangles with edge length b as $n \to +\infty$.

3. $\mathbb{R}^m$ **is nicely triangulable.** We start with a cubical regular grid and define T as the barycentric subdivision of that grid; see Figure 8, right. Precisely, for each cell in the grid, we insert one vertex at its centroid. So each edge is subdivided into 2 edges sharing the inserted vertex. We then recursively subdivide the cells by ascending dimension. Each cubical k-cell has $2k$ cubical $(k-1)$-cells on its boundary. We subdivide each k-cell as a cone whose apex is the inserted vertex and whose basis is the subdivided boundary of that cell. We claim that all stars in T are convex. Indeed each vertex in T is the centroid of an initial cubical cell of dimension between 0 and m. Consider the vertex v that was inserted at the center of the k-dimensional cubical cell D_v and let us describe the set of vertices V_v in the link of v in T. The vertices of V_v can be partitioned in two subsets. The first subset contains vertices in the k-flat that supports D_v while the second subset contains vertices in the $(d-k)$-flat passing through v and orthogonal to the k-flat supporting D_v. The vertices in the first subset lie on the boundary of D_v and the vertices in the second subset lie on the boundary of a $(m-k)$-cube. Since

in both flats of respective dimension k and $m-k$ the vertices in V_v are in convex position, it results that vertices in V_v are in convex position in $\mathbb{R}^m$. As a result, it can be proved (details are skipped) that the star of v is the convex hull of V_v. Finally, we let h be the identity map and n the inverse of the size of the grid. □

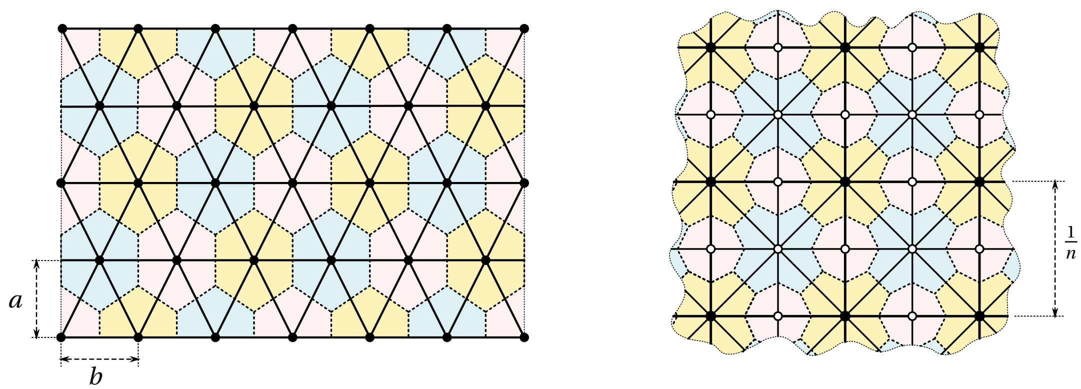

Figure 8: Triangulating $\mathbb{T}^2$ (left) and $\mathbb{R}^2$ (right).

8 Conclusion

In this paper, we introduce the notion of nicely triangulable shapes. We prove that the Čech complex of a point set at scale α can be simplified by collapses into a triangulation of a nicely triangulable shape, assuming the point set samples well enough the shape and the scale parameter α has been well chosen. Precisely, the sampling conditions and values of α are exactly the same as the ones required by the reconstruction theorem of Niyogi, Smale and Weinberger [15] for guaranteeing that the α-offset of a point set deformation retracts to the shape that the points sample. We conclude with a few open questions:

(1) Our result assumes the shape to be nicely triangulated. In this paper, we list a few simple spaces which enjoy this property. The list includes all surfaces $C^{1,1}$ diffeomorphic to a sphere. Is it possible to extend this list to a larger class of manifolds? In particular, we conjecture that surfaces embedded in $\mathbb{R}^3$ with a positive reach are nicely triangulable.

(2) Unfortunately, our proof is not constructive. Indeed, the order in which to collapse faces is determined by sweeping space with a t-offset of the shape for decreasing values of t. Since the common setting consists in describing the shape through a finite sample, the knowledge of the t-offsets of the shape is lost. Nonetheless, is it possible to turn our proof into an algorithm?

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A Two geometric lemmas

The following lemma is useful for establishing Lemma 3.

Lemma 12. *Let $X \subset \mathbb{R}^d$ be a non-empty compact set and $B(c,\rho)$ its smallest enclosing ball. For all points $z \in \mathbb{R}^d$ and all real numbers $\alpha \geq \rho$, the following implications hold*

(i) $X \subset \mathrm{B}(z,\alpha) \implies \mathrm{B}(c,\alpha - \sqrt{\alpha^2-\rho^2}) \subset \mathrm{B}(z,\alpha)$;

(ii) $\mathrm{B}(c,\alpha - \sqrt{\alpha^2-\rho^2}) \subset \mathrm{B}^\circ(z,\alpha) \implies X \cap \mathrm{B}^\circ(z,\alpha) \neq \emptyset$.

Proof. To establish (i), assume for a contradiction that $\mathrm{B}(z,\alpha)$ does not contain $\mathrm{B}(c,\alpha-\sqrt{\alpha^2-\rho^2})$ or equivalently that $\|c-z\| > \sqrt{\alpha^2-\rho^2}$. This implies that the smallest ball enclosing $\mathrm{B}(c,\rho)\cap\mathrm{B}(z,\alpha)$ has radius less than ρ. Since this intersection contains X, this would contradict the definition of ρ as the radius of the smallest ball enclosing X.

It is not hard to check (by contradiction) that the center of the smallest ball enclosing X lies on the convex hull of X. It follows that for any half-space H whose boundary passes through c, the intersection $X \cap H \cap B(c,\rho)$ is non-empty. To establish (ii), we may assume that $c \neq z$ for otherwise the result is clear. Let H be the half-space containing z whose boundary passes through c and is orthogonal to the segment cz; see Figure 9, left. If $\mathrm{B}(c,\alpha-\sqrt{\alpha^2-\rho^2}) \subset \mathrm{B}^\circ(z,\alpha)$, then $H\cap B(c,\rho) \subset \mathrm{B}^\circ(z,\alpha)$. Since $X\cap H\cap B(c,\rho)\neq\emptyset$, it follows that $X\cap\mathrm{B}^\circ(z,\alpha)\neq\emptyset$. □

The following lemma is useful for establishing Lemma 11.

Lemma 13. *Let $X \subset \mathbb{R}^d$ be a non-empty compact set and $B(c,\rho)$ its smallest enclosing ball. For all α and δ such that $\alpha \geq \rho$ and $\alpha - \sqrt{\alpha^2-\rho^2} \leq \delta$, the following inclusion holds:* $\mathrm{Hull}_\alpha(X) \subset [\mathrm{Conv}(X)]^{\oplus\delta}$.

Proof. See Figure 9, right. Consider a unit vector $u \in \mathbb{S}^{d-1}$ and let L_u be the half-line emanating from c in direction u. Let B^u_α denote the ball with radius α centered on L_u containing X and whose center is furthest away from c. By construction, X is contained in the intersection of the two balls B^u_α and $B(c,\rho)$. The boundary of $B^u_\alpha \cap B(c,\rho)$ consists of two spherical caps and we let C^u_α be the one lying of the sphere bounding B^u_α. Observe that X has a non-empty intersection with C^u_α and for all $\beta\geq\alpha$, the ball B^u_β intersects C^u_α. The largest distance between a point of C^u_α and B^u_β is upper bounded by the height of C^u_α which is less than of equal to $\alpha - \sqrt{\alpha^2-\rho^2} \leq \delta$. We thus get that $B^u_\alpha \subset [B^u_\beta]^{\oplus\delta}$. Considering this inclusion over all directions u for $\beta = +\infty$ yields the result. □

B $C^{1,1}$ diffeomorphism preserves nicely triangulable manifolds

The goal of this section is to prove Theorem 5. Let us provide some additional definitions. A $C^{1,1}$ function is a differentiable function with a Lipschitz derivative. A $C^{1,1}$ structure on a manifold is an equivalence class of atlases whose transition functions are $C^{1,1}$. Finally, $C^{1,1}$ diffeormorphisms betwen $C^{1,1}$ manifolds are defined accordingly. In Theorem 5, we restrict our attention to shapes which are $C^{1,1}$ compact manifolds without boundary embedded in $\mathbb{R}^d$ and whose embedding are themselves $C^{1,1}$ (for the natural smooth structure of $\mathbb{R}^d$). We will say that such shapes are compact $C^{1,1}$ manifolds embedded in $\mathbb{R}^d$ for short. A compact manifold embedded in $\mathbb{R}^d$ is $C^{1,1}$ if and only if it has a positive reach [14].

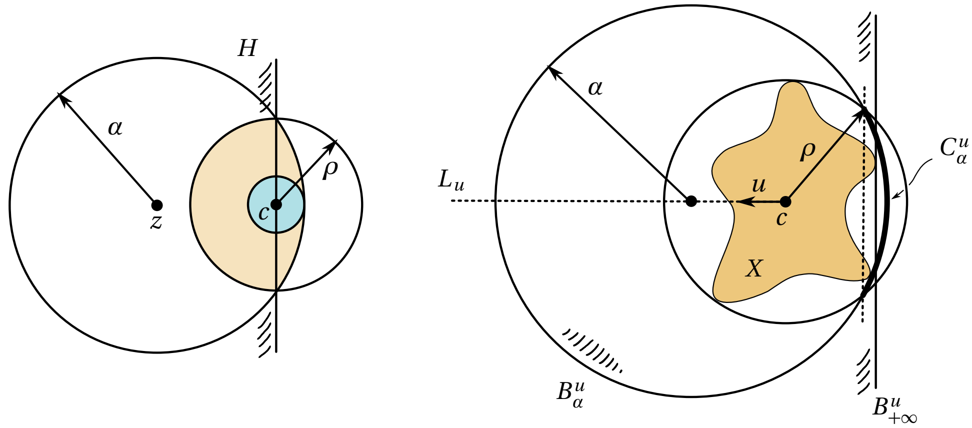

Figure 9: Notation for the proof of Lemma 12 (left) when $\|c-z\| = \sqrt{\alpha^2-\rho^2}$ and for the proof of Lemma 13 (right).

Proof of Theorem 5. Let $x \in M$. Since M is a compact $C^{1,1}$ k-manifold embedded in $\mathbb{R}^d$, there exists a k-dimensional affine space $T_M(x) \subset \mathbb{R}^d$ tangent to M at x. Let $\pi_x : M \to T_M(x)$ be the orthogonal projection onto the tangent space $T_M(x)$ and let $\pi'_{\Phi(x)} : M' \to T_{M'}(\Phi(x))$ the orthogonal projection onto $T_{M'}(\Phi(x))$. Since M and M' are compact, we can find two constants K and K' independent of x such that:

$$\forall y \in M, \quad \|y - \pi_x(y)\| < K\|y-x\|^2 \tag{3}$$
$$\forall y \in M', \quad \|y - \pi'_{\Phi(x)}(y)\| < K'\|y-\Phi(x)\|^2 \tag{4}$$

Given $t_0 > 0$, we consider the open set $U_x = M \cap \mathrm{B}^\circ(x, t_0)$ and adjust t_0 in such a way that

1. The restriction $\pi_x : U_x \to \pi_x(U_x)$ is an homeomorphism for all $x \in M$;
2. The restriction $\pi'_{\Phi(x)} : \Phi(U_x) \to \pi'_{\Phi(x)}(\Phi(U_x))$ is also an homeomorphism for all $x \in M$.

The collection of pairs $\{(U_x, \pi_x)\}_{x\in M}$ forms an atlas in the $C^{1,1}$ structure of M. Similarly, the collection of pairs $\{(\Phi(U_x), \pi'_{\Phi(x)})\}_{x\in M}$ forms an atlas in the $C^{1,1}$ structure of M'. Let $\mathscr{T}_M(x)$ be the linear space associated to $T_M(x)$ and denote by $D\Phi_x$ the derivative of Φ at x, seen as a linear map between $\mathscr{T}_M(x)$ and $\mathscr{T}_{M'}(\Phi(x))$. Since M is compact, there is a constant K_Φ independent of x such that:

$$\forall y \in M, \quad \|\Phi(y) - \Phi(x) - D\Phi_x(\pi_x(y) - x)\| < K_\Phi \|y-x\|^2, \tag{5}$$

and two positive numbers $\kappa_2 \geq \kappa_1 > 0$, again independent of x by compactness of M, such that

$$\forall u \in \mathscr{T}_M(x), \quad \kappa_1\|u\| \leq \|D\Phi_x(u)\| \leq \kappa_2\|u\|. \tag{6}$$

Consider the affine function $\hat\Phi_x : T_M(x) \to T_{M'}(\Phi(x))$ defined by $\hat\Phi_x(y) = \Phi(x) + D\Phi_x(y-x)$. Combining Equations (3) (4) and (5), we can find a constant L_Φ independent of x such that for all $t < t_0$ and all compact sets $A \subset M \cap B(x,t)$:

$$d_H(\hat\Phi_x \circ \pi_x(A), \pi'_{\Phi(x)} \circ \Phi(A)) < L_\Phi t^2 \tag{7}$$

Now, assume that M is nicely triangulable and let us prove that M' is also nicely triangulable. By definition, we can find $\rho_0 > 0$ and $\eta_0 > 0$ such that, for all $0 < \rho < \rho_0$, there is a $(\rho, \eta_0\rho)$-nice

triangulation T of M with respect to some $(h,\mathscr{C})$. Suppose $\mathscr{C} = \{C_v \mid v \in V\}$ and consider the covering $\mathscr{C}' = \{\Phi(C_v) \mid v \in V\}$, the homeomorphism $h' = \Phi \circ h : |T| \to M'$, the real numbers $\rho' = 2\kappa_2\rho$ and $\eta'_0 = \frac{\kappa_1\eta_0 - 5L_\Phi\rho}{2\kappa_2}$. Let us prove that by choosing ρ small enough, T is a $(\rho',\eta'_0\rho')$-nice triangulation of M' with respect to $(\mathscr{C}',h')$. In other words, we need to check that conditions (ii) and (iii) of Definition 2 are satisfied for $\mathscr{C} = \mathscr{C}'$, $h = h'$, $\rho = \rho'$ and $\delta = \eta'_0\rho'$. Take $v \in V$ and set $x = h(v)$, $C = C_v$, $S = h(\mathrm{St}_T(v))$.

(ii) By definition of T, we have $x \in C \subset \mathrm{B}^\circ(x,\rho)$. Taking the image of this relation under Φ and choosing $\rho > 0$ small enough, we get that $\Phi(x) \in \Phi(C) \subset \Phi(\mathrm{B}^\circ(x,\rho)) \subset \mathrm{B}^\circ(\Phi(x),\rho')$. The last inclusion is obtained by combining Equations (3), (5) and (6).

(iii) Let us choose a positive real number $\rho < \min\{\rho_0, \frac{t_0}{2}\}$ small enough to ensure that $\eta'_0 > 0$ and let us prove that $M' \cap [\mathrm{Conv}(C)]^{\oplus\eta'_0\rho'} \subset S$. By choice of T as a $(\rho,\eta_0\rho)$-nice triangulation of M with respect to $(h,\mathscr{C})$, we have that $M \cap \mathrm{Conv}(C)^{\oplus\eta_0\rho} \subset S$. Furthermore, $C \subset B(x,\rho)$ and $S \subset B(x,2\rho)$. Thus, by choosing $\rho < \frac{t_0}{2}$, we have $S \subset U_x$ and

$$U_x \cap \mathrm{Conv}(C)^{\oplus\eta_0\rho} \subset S.$$

Taking the image by the homeomorphism $\pi_x : U_x \to \pi_x(U_x)$ on both sides and using $\pi_x(A \cap B) = \pi_x(A) \cap \pi_x(B)$ we get

$$\pi_x(\mathrm{Conv}(C)^{\oplus\eta_0\rho}) \subset \pi_x(S).$$

Let $B_k(0,r)$ denote the k-dimensional ball of $\mathscr{T}_M(x)$ centered at the origin with radius r. Writing $A \oplus B = \{a + b \mid a \in A, b \in B\}$ for the Minkowski sum of A and B, it is not too difficult to prove that $\pi_x(A^{\oplus\delta}) = \pi_x(A) \oplus B_k(0,\delta)$. It follows that

$$\pi_x(\mathrm{Conv}(C)) \oplus B_k(0,\eta_0\rho) \subset \pi_x(S).$$

Taking the image under $\hat{\Phi}_x$ on both sides we get

$$\hat{\Phi}_x \circ \pi_x(\mathrm{Conv}(C)) \oplus D\Phi_x B_k(0,\eta_0\rho) \subset \hat{\Phi}_x \circ \pi_x(S)).$$

Let $B'_k(0,r)$ denote the k-dimensional ball of $\mathscr{T}_{M'}(\Phi(x))$ centered at the origin with radius r. Using Equation (6) we get that $B'_k(0,\kappa_1\eta_0\rho) \subset D\Phi_x B_k(0,\eta_0\rho)$. Since $\hat{\Phi}_x$ and π_x are both affine, so is the composition and therefore $\hat{\Phi}_x \circ \pi_x(\mathrm{Conv}(C)) = \mathrm{Conv}(\hat{\Phi}_x \circ \pi_x(C))$. It follows that

$$\mathrm{Conv}(\hat{\Phi}_x \circ \pi_x(C)) \oplus B'_k(0,\kappa_1\eta_0\rho) \subset \hat{\Phi}_x \circ \pi_x(S).$$

Recalling that $C \subset B(x,\rho)$ and $S \subset B(x,2\rho)$ and combining the above inclusion with Equation (7) we obtain

$$\mathrm{Conv}(\pi'_x \circ \Phi(C)) \oplus B_k(0,\kappa_1\eta_0\rho - 5L_\Phi\rho^2) \subset \pi'_x \circ \Phi(S).$$

Interchanging Conv and π'_x, noting that $\eta'_0\rho' = \kappa_1\eta_0\rho - 5L_\Phi\rho^2$ and using $\pi'_x(A^{\oplus\delta}) = \pi'_x(A) \oplus B_k(0,\delta)$ we get

$$\pi'_x(\mathrm{Conv}(\Phi(C))^{\oplus\eta'_0\rho'}) \subset \pi'_x \circ \Phi(S).$$

Since $\pi'_x : \Phi(U_x) \to \pi'_{\Phi(x)}(\Phi(U_x))$ is homeomorphic, we thus obtain $M' \cap \mathrm{Conv}(\Phi(C))^{\oplus\eta'_0\rho} \subset \Phi(S)$ as desired. $\square$

C Getting rid of genericity assumptions

In the proof of Lemma 10, we consider a 1-parameter family of nerves, $K_t = \mathrm{Nrv}\{D_p(t) \mid p \in P\}$ for $t \in [0,1]$. Letting Δ_t be the set of simplices that disappear at time t, we assume that the following generic condition is satisfied:

($\star$) If $\Delta_t \neq \emptyset$ for some $t \in [0,1)$, then Δ_t has a unique inclusion-minimal element $\sigma_{\min}$.

If not the case, the idea is to apply a small perturbation to the family $\mathscr{D} = \{D_p(t) \mid p \in P\}$ which will leave unchanged K_0 and K_1 and such that after perturbation (1) $\mathscr{D}$ will still satisfy the hypotheses of the Lemma 10; (2) the generic condition ($\star$) will hold.

Definition 4. *Two simplices σ_1 and σ_2 are said to be in conjunction at time t if they are both inclusion-minimal elements of Δ_t for some $t \in [a,b)$.*

Consider two simplices that are in conjunction at time t, say σ_1 and σ_2. Suppose $p \in \sigma_1$ and $p \notin \sigma_2$. Consider an increasing bijection $\psi : [0,1] \to [0,1]$ that differs from identity only in a small neighborhood of t that does not include any other event times. By replacing D_p by $D_p \circ \psi$, we change the time at which σ_1 disappears while keeping unchanged the time at which σ_2 disappears. After this operation, σ_1 and σ_2 are not in conjunction anymore. Furthermore, the operation does not create any new pair of simplices in conjunction. By repeating this operation a finite number of times, we thus get a new collection as required.

The generic condition in the proof of Theorem 2 is removed in a similar way.